Extended self-similarity in the numerical simulation of three-dimensional homogeneous flows

M. Briscolini,[1] P. Santangelo,[1] S. Succi,[1] and R. Benzi[2]
[1]*IBM European Center for Scientific and Engineering Computing, via Oceano Pacifico 171, 00144 Roma, Italy*
[2]*Dipartimento di Fisica, Università di Roma "Tor Vergata," via della Ricerca Scientifica 1, 00173 Roma, Italy*
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We report the statistical analysis of several velocity configurations obtained by performing numerical simulations of three-dimensional homogeneous incompressible turbulence. This analysis provides full support to the idea, recently proposed by Benzi *et al.* [Phys. Rev. E **48**, 29 (1993)], that fluid flows exhibit extended self-similarity, a sort of generalized scale invariance, which holds at high as well as at low-to-moderate Reynolds numbers.

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The main parameter controlling the physics of turbulent flows is the Reynolds number defined as

$$\mathrm{Re}=\frac{uL}{\nu}\,, \tag{1}$$

where u is the characteristic flow speed, L is the characteristic macroscale of the flow, and ν is the fluid kinematic viscosity.

In the limit of infinite Reynolds number (vanishing viscosity), the Navier-Stokes equations are invariant under the following group of scaling transformations for the space x, time t, and velocity v variables:

$$x\rightarrow\lambda x, \quad v\rightarrow\lambda^h v, \quad t\rightarrow\lambda^{1-h}t, \tag{2}$$

where h is an arbitrary exponent labeling the invariance group.

Scaling lies at the heart of Kolmogorov's 1941 theory (K41) [1], which is based on the following assumptions.

(i) In the limit of Reynolds number tending to infinity the energy dissipation rate ϵ remains finite.

(ii) The energy flux from large scales, where turbulence is produced, to small scales, where it is dissipated, is scale independent.

(iii) In the limit of infinite Reynolds numbers, scale invariance is restored at least in a statistical sense.

It is readily checked that the K41 assumptions remove any arbitrariness in the choice of the exponent h, which is forced to the value $h=\frac{1}{3}$.

By dimensional arguments, the energy flux at scale r can be written as

$$\epsilon_r\sim\langle\delta v_r^3\rangle/r, \tag{3}$$

where $\delta v_r=v_x(\mathbf{x}+r)-v_x(\mathbf{x})$ is the longitudinal velocity difference component at the scale separation r oriented according to the x direction and the brackets denote ensemble averaging. Owing to the second K41 assumption, one derives $\langle\delta v_r\rangle\sim r^{1/3}$, i.e., $h=\frac{1}{3}$. In general, one can define the scaling exponents ζ_p of the structure functions $S_p(r)$ by the following relation:

$$S_p(r)\equiv\langle\delta v_r^p\rangle\sim r^{\zeta_p} \tag{4}$$

and, for the K41 theory, one has

$$\zeta_p=\frac{p}{3}\,. \tag{5}$$

The aforelisted equations should apply only to the range of scales where self-similarity is not broken either by boundary conditions (large scales) or by molecular dissipation. This defines the so-called inertial regime

$$\eta\ll r\ll L, \tag{6}$$

where L is the integral scale of motion and $\eta=\nu^{3/4}\epsilon^{-1/4}$ is the dissipative length, i.e., the typical scale at which dissipation takes over nonlinear transfer in controlling the dynamics of the system.

There is a wide body of experimental evidence that Kolmogorov scaling is violated in the inertial range of high-Reynolds-numbers flows [2–4]. The scaling exponents ζ_p become significantly smaller in respect to the ones predicted by the K41 linear law $\zeta_p=p/3$ as the parameter p is increased. This is interpreted as the statistics of small scales (the one dictating the behavior of higher-order momenta) becoming increasingly non-Gaussian, a phenomenon usually referred to as to "intermittency."

From a practical point of view, the inertial region is defined by the range of scales where the third-order structure function follows the K41 law

$$S_3=-\tfrac{4}{5}\,\epsilon r. \tag{7}$$

The larger the Reynolds number, the broader the inertial region. Unfortunately, for the low-to-moderate Reynolds numbers accessible to direct numerical resolution, this range is often very narrow.

Very recently, an alternative way of testing scaling properties in flow turbulence has been proposed. The basic idea is to represent the quantity S_p no longer explicitly related to the space separation, but rather as a function of the third-order structure function S_3 [4]. The rationale behind this idea is that the behavior of S_p is dictated solely by the third-order structure function S_3 via a set of scaling exponents α_p according to the following relation:

$$S_p\sim S_3^{\alpha_p}. \tag{8}$$

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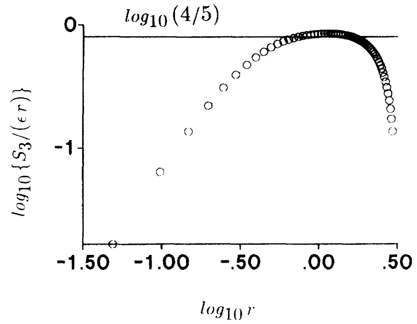

FIG. 1. The plot of $\log_{10}[S_3/(\epsilon r)]$ versus $\log_{10} r$. The continuous line is the $\log_{10}(\frac{4}{5})$ value predicted by the K41 formula (7) in correspondence of scales within the inertial range.

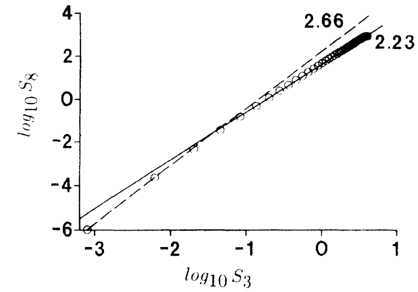

FIG. 3. The plot of the eighth-order structure function $\log_{10} S_8$ versus $\log_{10} S_3$. The plot reports the line with slope 2.23 which represents the least-square linear fit of $\log_{10} S_8$ in the range 5η–32η.

This relation represents a generalization of the concept of scale invariance as expressed by formula (4) [whence the definition of extended self-similarity (ESS)] as it becomes apparent once S_3 is identified with a generalized space coordinate. This generalized coordinate reduces to the usual spatial separation r only in the limit of fully developed turbulence (with r within the inertial range). In this limit, the notion of ESS reduces to the ordinary picture of self-similarity underlying the concept of scale invariance. The crucial point about ESS is that the relation (8) applies to fully developed turbulence as well as to low-to-moderate turbulence where Eq. (4) is not applicable. Moreover, relation (8) holds for a much wider range of scales than (4), almost down to the dissipative scale η. Experimental data seem to indicate that the ESS scaling exponents α_p are practically the same as those observed in fully developed turbulence. In other words, not only does ESS reveal a much more extended "generalized" scaling region, but it also indicates an unexpected (and welcome) feature of turbulence: intermittency corrections to K41 scaling are already visible at low Reynolds number, and are in a quantitative match those observed at high Reynolds numbers.

This would suggest that concepts such as scaling and intermittency do not exclusively belong to fully developed turbulence but preserve their meaning also in the case of moderately low turbulence, provided the right gauge S_3 instead of r is used to define them.

ESS might have far-reaching consequences for the understanding and modeling of fluid turbulence; in particular, it

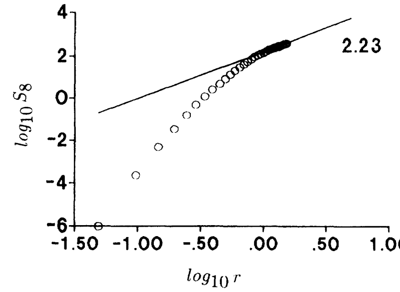

FIG. 2. The plot of the eighth-order structure function $\log_{10} S_8$ versus $\log_{10} r$.

may bring part of the physics of fully developed flows within reach of the present-day capabilities of computer simulation.

Under these conditions, the task of accumulating experimental and numerical data to corroborate or disclaim the ESS assumption appears of primary importance.

In this paper, we bring further evidence of ESS, by showing that it applies also to the case of three-dimensional simulation of low-Reynolds-number homogeneous incompressible turbulence. To this purpose, we have performed a pseudospectral numerical simulation [5], running in-core on an IBM RS/6000 model 560 workstation computer. The simulation is performed in a box of size 2π discretized by 128^3 grid points which corresponds to a grid spacing $\Delta x \simeq 0.05$. The viscosity and the time step are $\nu = 0.01$ and $\Delta t = 0.005$, respectively. The initial condition is randomly generated according to the Gaussian distribution and the $k^{-5/3}$ energy spectrum; the total initial energy is scaled to be $E(0) = 0.5$ and, consequently, the rms velocity is $v_{\rm rms} = [2/3E(0)]^{1/2} \simeq 0.6$. This corresponds to $\mathrm{Re}_\lambda = v_{\rm rms}\lambda/\nu \simeq 38$ with $\lambda = (15\nu v_{\rm rms}/\epsilon)^{1/2} \simeq 0.7 \sim 14\Delta x$.

The turbulence is forced by imposing that modes in the shell $k \leq 1$ do not evolve in time [6]. This kind of forcing allows us to perform a long time simulation preserving the dynamical equilibrium between the energy enforced at large scales and the one dissipated at smaller ones. This has been confirmed by plotting the mean dissipation rate ϵ in time and observing that, after an initial transient stage, ϵ undergoes very small fluctuations around the mean value $\epsilon \simeq 0.2$ so that the corresponding Kolmogorov scale is forced to be $\eta \simeq \Delta x$ for the entire time lapse of simulation.

The simulation has advanced 20 000 steps which correspond to about 10 macroscale eddy turnover times estimated as $\tau_0 \sim 2\pi/v_{\rm rms} \simeq 10$. To enrich the statistics, 40 velocity configurations have been saved each 500 time steps in correspondence to a time interval equal to $\tau_0/4$, then sufficiently large to ensure the statistical independence among different velocity configurations. The final statistical ensemble to be processed is composed of about 10^8 data which allows us to estimate with adequate accuracy the statistics of momenta at least for $p \leq 10$ [2,3,6] as the criteria adopted in Ref. [6] for even-order momenta suggest. For the following discussion, we have compared our numerical estimate of $S_3 = |\langle \delta u_r^3 \rangle|$, which directly comes from the definition (4), with the mean of absolute values ($S_3^* = \langle |\delta u_r|^3 \rangle$) which are more conve-

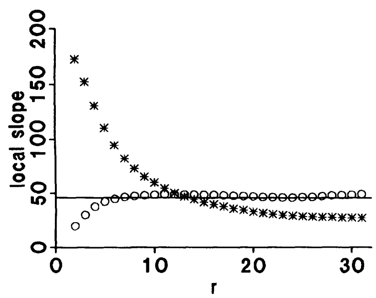

FIG. 4. The local slope $d\log_{10}S_8/d\log_{10}S_3^{2.23}$ and $d\log_{10}S_8/d\log_{10}S_3^{8/3}$ versus r/η which evidence the range of scales where ESS provides the 2.23 linear slope for the $\log_{10}S_8$ structure function.

nient in order to reduce possible fluctuations induced by the reduced set of data over which to estimate the statistical momenta. We have conducted a qualitative analysis, similar to that adopted in Ref. [7], by plotting $\log_{10}S_3$ versus $\log_{10}S_3^*$ verifying that in the present case the relationship $\log_{10}S_3 \simeq \log_{10}S_3^*$ is well respected for the whole range of dynamical scales resolved in our simulation. As a consequence, we decided to use S_3^*, instead of the more correct S_3, for our analysis and, in the following, we suppress the symbol * in the definition.

As a first issue, we verify the extension of the inertial range reproduced by our simulation reporting in Fig. 1 the plot of $\log_{10}[S_3/(\epsilon r)]$ as a function of $\log_{10}r$. From this figure, a narrow plateau is visible, ranging from $r=25$ to $r=35$ grid units, with a plateau height in excellent agreement with theoretical prediction $\log_{10}(\frac{4}{5})$ provided by formula (7).

Next, we consider higher-order structure functions up to $p=10$. In particular, we focus our attention on the eighth-order structure function S_8. This quantity is log-log plotted as a function of r in Fig. 2. Again, a (questionable) scaling region is visible only in a very narrow range of values of r.

Let us now consider the same data using ESS. To this purpose, we plot $\log_{10}S_8$ versus $\log_{10}S_3$. This information is displayed in Fig. 3, from which a strikingly wide scaling region, spanning most of the computational resolution, is clearly visible. The scaling exponent is 2.23, in very close agreement with experimental data.

The close match with experimental data is further demonstrated in Fig. 4, where the local slopes $d\log_{10}S_8/d\log_{10}S_3^{2.23}$ and $d\log_{10}S_8/d\log_{10}S_3^{8/3}$ are reported as a function of r/η. From this figure, we deduce robust evidence that (i) ESS does apply to moderate-Reynolds-number three-dimensional homogeneous incompressible turbulence, (ii) the scaling exponents measured with ESS are in quantitative agreement with experimental data accounting for intermittency, and (iii) the scaling region extends to scales as small as $\sim 5\,\eta$.

TABLE I. The ζ_p exponent structure functions experimentally and numerically estimated as compared to the ESS measured α_p exponents.

p	α_p	ζ_p [2]	ζ_p [4]	ζ_p [9]
2	0.69	0.71	0.71	0.69
4	1.29	1.33	1.28	1.28
6	1.80	1.80	1.78	1.76
8	2.23	2.22	2.22	2.14
10	2.58	2.59	2.60	2.48

These conclusions are corroborated by further analysis at various values of the parameter p. A list of the measured scaling exponents is given in Table I. This confirms previous findings according to which the scaling exponents associated with ESS are practically the same as those computed for fully developed homogeneous turbulent flows.

From the practical point of view, we feel that the most important outcome of ESS is the extension of the lowest end of the scaling region down to about ~ 5 dissipative lengths, i.e., a factor $\sim \lambda/5\eta \sim 3-4$ smaller than the same quantity under "conventional" analysis. This substantiates our previous assertion according to which ESS brings the physics of turbulence scaling within reach of present-day computers.

This paper has evidenced that ESS is applicable for homogeneous flows in the range of low-to-moderate Reynolds numbers. Finally, we want to comment about a possible discrepancy one might observe between the results discussed so far and those reported in Ref. [8] where ESS was verified only for $r \geq 25\,\eta$ for moments of order 8 and higher. In Ref. [8] the experimental results refer to measurements taken close to a boundary layer, i.e., in a nonhomogeneous flow. As discussed in detail in Ref. [7], going from a homogeneous to a nonhomogoneous flow, the range where ESS is verified reduces and the scaling exponents decrease. The effects of nonhomogeneity and large scale shear flows on the scaling exponents and ESS scaling range are presently under investigation.

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